JEE(ADVANCED)-2013 PAPER 1

MATHEMATICS

41. Sol: (D)

Any point B on line is $(2\lambda-2,-\lambda-1,3\lambda)$

Point B lies on the plane for some λ.

$\Rightarrow (2\lambda-2)+(-\lambda-1)+3\lambda=3$

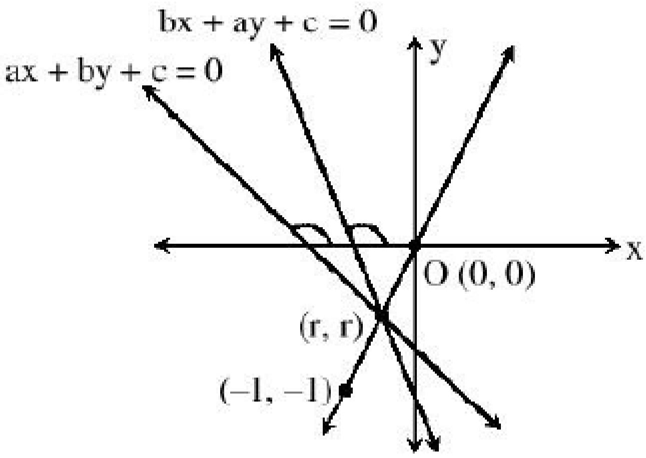
$\Rightarrow 4\lambda=6 \Rightarrow \lambda\frac{3}{2} \Rightarrow B=\left(1,\frac{-7}{2},\frac{5}{2}\right)$

The foot of the perpendicular from point $(-2,-1,0)$ on the plane is the point $A(0,1,2)$

$\Rightarrow D.R.\ of\ AB=\left(1,\frac{-7}{2},\frac{5}{2}\right)=(2,-7,5)$

Hence $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

42. Sol: (A)



For point of intersection $(a-b)x_1=(a-b)y_1$

$\Rightarrow$ point lie on line $y=x \quad \ldots(1)$

Let point is (r,r)

$\sqrt{(r-1)^2+(r-1)^2}<2\sqrt{2}$

$\sqrt{2}|r-1|<2\sqrt{2}$

$\Rightarrow |r-1|<2$

$\Rightarrow -1<r<3$

$\Rightarrow (-1,-1)$ lies on the opposite side of origin for both

$\Rightarrow -a-b+c<0$

$\Rightarrow a+b-c>0$

43. Sol: (B)

$$y_1 = \sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

$$y_2 = s\sqrt{2}\left|\sin\left(\frac{\pi}{4}-x\right)\right|$$

$$\Rightarrow Area = \int_0^{\frac{\pi}{4}}\left((\sin x+\cos x)-(\cos x-\sin x)\right)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left((\sin x+\cos x)-(\sin x-\cos x)dx\right)$$

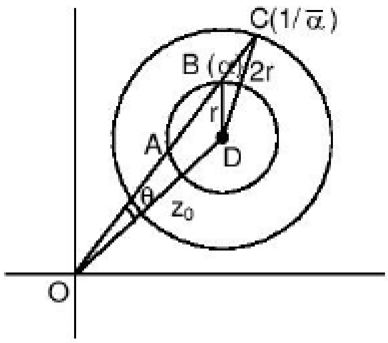
$$= 4-2\sqrt{2}$$

44. Sol: (A)

P (at least one of them solves correctly) $=1-P$ (none of them solves correctly)

$$=1-\left(\frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}\times\frac{7}{8}\right)=\frac{235}{256}$$

45. Sol: (C)



$OB = |\alpha|$

$$OC = \frac{1}{|\alpha|} = \frac{1}{|\alpha|}$$

In ΔOBD

$$\cos\theta = \frac{|z_0|^2+|\alpha|^2-r^2}{2|z_0||\alpha|}$$

$In\ \Delta OCD$

$$\cos\theta = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$$

$$\frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|} = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$$

$$\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$$

46. Sol: (C)

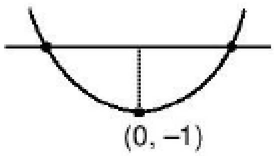
$Let\ f(x) = x^2 - x\sin x - \cos x \Rightarrow f'(x) = 2x - x\cos x$

$\lim_{x\to\infty} f(x) \to \infty$

$\lim_{x\to\infty} f(x) \to \infty$

$f(0) = -1$

Hence 2 solutions



47. Sol: (D)

$Given\ f'(x) - 2f(x) < 0$

$\Rightarrow f(x) < ce^{2x}$

Put, $x = \frac{1}{2} \Rightarrow c > \frac{1}{e}$

Hence, $f(x) < ce^{2x-1}$

$$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} ce^{2x-1}dx$$

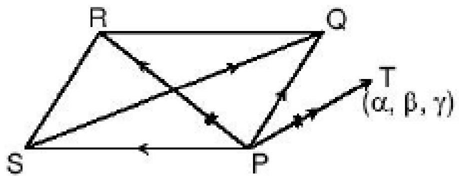
$$0 < \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$$

48. Sol: (C)

$$\text{Area of base } PQRS = \frac{1}{2}\left|\overrightarrow{PR}\times\overrightarrow{SQ}\right| = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & -4\end{vmatrix}$$

$$\text{Height} = \text{proj.of } PT \text{ on } \hat{i}-\hat{j}+\hat{k} = \left|\frac{1-2+3}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

$$\text{Volume} = \left(5\sqrt{3}\right)\left(\frac{2}{\sqrt{3}}\right) = 10\,\text{cu. units}$$



49. Sol: (A)

$$\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(n^2+n+1\right)\right)$$

$$\cot\left(\sum_{n=1}^{23}\tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right)\right)$$

$$\Rightarrow \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right) = \frac{25}{23}$$

50. Sol: (B, D)

$$\text{The curve passes through}\left(1,\frac{\pi}{6}\right)$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = in\,x + \frac{1}{2}$$

51. Sol: (B, C)

$$\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}.let\ y = vx$$

$$\Rightarrow \frac{dv}{\sec v} = \frac{dx}{x}$$

$$\int \cos\ vdv = \int\frac{dx}{x}$$

$\sin v = in\, x + c$

$\sin\left(\dfrac{y}{x}\right) = in\, x + c$

The common perpendicular is along $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 2\hat{i} + 3\hat{j} - 2\hat{k}$

$Let\ M \equiv (2\lambda, -3\lambda, 2\lambda)$

$So, \dfrac{2\lambda - 3}{1} = \dfrac{-3\lambda + 1}{2} = \dfrac{2\lambda - 4}{2} \Rightarrow \lambda = 1$

$So, M \equiv (2, -3, 2)$

Let the required point be P

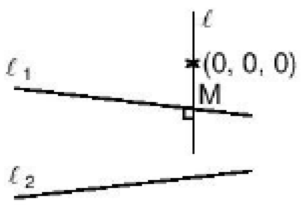
Given that PM $= \sqrt{7}$

$\Rightarrow (3 + 2s - 2)^2 + (3 + 2s + 3)^2 + (3 + s - 2)^2 = 17$

$\Rightarrow 9s^2 + 28s + 20 = 0$

$\Rightarrow s = -2, -\dfrac{10}{9}$

$So, P \equiv (-1, -1, 0)\ or \left(\dfrac{7}{9}, \dfrac{7}{9}, \dfrac{8}{9}\right)$



52. Sol: (A, D)

$We\ have\ f'(x) = \sin \pi x + \pi x\ cos \pi x = 0$

$\Rightarrow \tan \pi x = -\pi x$

$\Rightarrow \pi x \in \left(\dfrac{2n+1}{2}\pi, (n+1)\pi\right) \Rightarrow x \in \left(n + \dfrac{1}{2}, n+1\right) \in (n, n+1)$

53. Sol: (C, D)

$S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k-1)}{2}} k^2 = \sum_{r=0}^{(n-1)} \left((4r+4)^2 (4r+3)^2 - (4r+2)^2 - (4r+1)^2\right)$

$= \sum_{r=0}^{(n-1)} \left(2(8r+6) + 2(8r+1)\right)$

$= \sum_{r=0}^{(n-1)} (32r+20)$

$= 16(n-1)n + 20n$

$= 4n(4n+1)$

$= \begin{cases} 1056 \ for \ n=8 \\ 1332 \ for \ n=9 \end{cases}$

54. Sol: (A, C)

(A) $\left(N^T MN\right)^T = \mathrm{N}^T\mathrm{M}^T\mathrm{TN} = \mathrm{N}^{\mathrm{T}}\mathrm{MN}$ is symmetric and is $\mathrm{N}^{\mathrm{T}}\mathrm{MN}$ if M is skew symmetric

(B) $(MN-NM)^T = N^T M^T - M^T N^T = NM - MN = -(MN-NM)$. So, $(MN-NM)$ is skew symmetric

(C) $(MN)^T = N^T M^T = NM \neq MN$ if M and N are symmetric. So, MN is not symmetric

(D) $(adj.M)(adj.N) = adj(NM) \neq adj(MN)$.

55. Sol: (A, C)

Let the sides of rectangle be $15k$ and $8k$ and side of square be x then

$(15k-2x)(8k-2x)$ is volume,

$v = 2\left(2x^3 - 23kx^2 + 60k^2 x\right)$

$\left.\frac{dv}{dx}\right|_{x=5} = 0$

$\left.6x^2 - 46kx + 60k^2\right|_{x=5} = 0$

$6x^2 - 23kx + 15 = 0$

$k = 3, \ k = \frac{5}{6}, Only \ k = 3$ is permissible

So, *the side are* 45 *and* 24

56. Sol: (5)

Let $(1,1,1),(-1,1,1),(1,-1,1),(-1,-1,1)$ be vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ rest of the vectors are $-\vec{a}, -\vec{b}, -\vec{c}, -\vec{d}$ and let us find the number of ways of selecting co-planar vectors.

Observe that out of any 3 coplanar vectors two will be collinear (anti parallel)

Number of ways of selecting the anti-parallel pair $= 4$

Number of ways of selecting the third vector $= 6$

Total $= 24$

Number of non co-planar selections ${}^8C_3 - 24 = 32 = 2^5, P = 5$

Alternative Solution:

Required value $=\dfrac{8\times6\times4}{3!}$

$\therefore P=5$

57. Sol: (6)

$Let\ P(E_1)=x, P(E_2)=y\ and\ P(E_3)=z$

$then\ (1-x)(1-y)(1-z)=p$

$x(1-y)(1-z)=\alpha$

$(1-x)y(1-z)=\beta$

$(1-x)(1-z)(1-z)=\gamma$

$So\ \dfrac{1-x}{x}=\dfrac{p}{\alpha}\quad x=\dfrac{\alpha}{\alpha+p}$

$similarly\ z=\dfrac{\gamma}{\gamma+p}$

$$so\ \frac{p(E_1)}{p(E_2)}=\frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}}=\frac{\frac{\gamma}{\gamma+p}}{\frac{\alpha}{\alpha+p}}=\frac{1+\frac{P}{\gamma}}{1+\frac{P}{\alpha}}$$

$$also\ given\ \frac{\alpha\beta}{\alpha-2\beta}=p=\frac{2\beta\gamma}{\beta-3\gamma}\Rightarrow\beta=\frac{5\alpha\gamma}{\alpha}$$

$$\text{Substituting back}\left(\alpha-2\left(\frac{5\alpha\gamma}{\alpha+4\gamma}\right)\right)p=\frac{\alpha.5\alpha\gamma}{\alpha+4\gamma}$$

$$\Rightarrow\alpha p-6p\gamma=5\alpha\gamma$$

$$\Rightarrow\left(\frac{p}{\gamma}+1\right)=6\left(\frac{p}{\alpha}+1\right)\Rightarrow\frac{\frac{p}{\gamma}+1}{\frac{p}{\alpha}+1}=6.$$

58. Sol: (6)

$Let\ T_{r-1}, T_r\ T_{r+1}$ are three consecutive terms of $(1+x)^{n+5}$

$T_{r-1}={}^{n+5}C_{r-2}(x)^{r-2}, T_r={}^{n+5}C_{r-1}x^{r-2}, T_{r+1}={}^{n+5}C_{r-1}x^r$

$Where,\ {}^{n+5}C_{r-2}:{}^{n+5}C_{r-1}:{}^{n+5}C_r=5:10:14$

$$\frac{{}^{n+5}C_{r-2}}{5}=\frac{{}^{n+5}C_{r-1}}{10}\Rightarrow n-3r=-3\quad\ldots(1)$$

$$\frac{{}^{n+5}C_{r-1}}{10}=\frac{{}^{n+5}C_r}{14}\Rightarrow 5n-12r=-30\quad\ldots(2)$$

From equation (1) *and* $(2)\, n=6$

59. Sol: (5)

$Clearly,\ 1+2+3+\ldots+n-2 \le 1224 \le 3+4+\ldots n$

$\Rightarrow \frac{(n-2)(n-1)}{2} \le 1224 \frac{(n-2)}{2}(3+n)$

$\Rightarrow n^2-3n-2446 \le 0\, and\, n^2+n-2454 \ge 0$

$\Rightarrow 49 < n < 51 \Rightarrow n=50$

$\therefore \frac{n(n+1)}{2}-(2k+1)=1224 \Rightarrow k=25 \Rightarrow k-20=5$

60. Sol: (9)

$\frac{x^2}{4}+\frac{y^2}{3}=1$

$y=\frac{\sqrt{3}}{2}\sqrt{4-h^2}\ at\ x=h$

Let $R(x_1,0)$

PQ is chord of contact, so $\frac{xx_1}{4}=1 \Rightarrow x=\frac{4}{x_1}$

Which is equation of $PQ, x=h$

So, $\frac{4}{x_1}=\eta \Rightarrow x_1=\frac{4}{h}$

$\Delta(h)=area\ of\ \Delta PQR=\frac{1}{2}PQ\times RT$

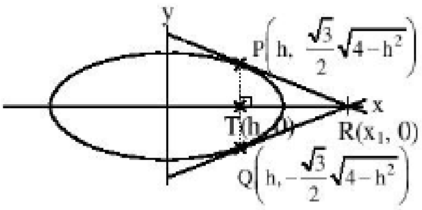
$=\frac{1}{2}\times\frac{2\sqrt{3}}{2}\sqrt{4-h^2}\times(x_1-h)=\frac{\sqrt{3}}{2h}(4-h^2)^{3/2}$

$\Delta'(h)=\frac{-\sqrt{3}(4+2h^2)}{2h^2}\sqrt{4-h^2}$ which is always decreasing

So, $\Delta_1=$ maximum of $\Delta(h)=\frac{45\sqrt{5}}{8}\ at\ h=\frac{1}{2}$

$\Delta_2=$ maximum of $\Delta(h)=\frac{9}{2}\ at\ h=1$

So, $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=\frac{8}{\sqrt{5}}\times\frac{45\sqrt{5}}{8}-8.\frac{9}{2}=45-36=9$


y
x
$P\left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2}\right)$
$T(h, 0)$
$R(x_1, 0)$
$Q\left(h, -\frac{\sqrt{3}}{2}\sqrt{4-h^2}\right)$